Lecture 6

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LOCAL LIE ALGEBRAS

I GENERAL LOCAL LIE ALGEBRAS

The motivating example Kirillov was trying to generalise was that of a Poisson bracket. From an algebraic point of view, Poisson brackets satisfy the same locality property as function multiplication (making the Poisson algebra manifestly local):

$$\{,\}: C^\infty(M)\times C^\infty(M) \to C^\infty(M) \quad : \quad \operatorname{supp}(\{f,g\}) \subset \operatorname{supp}(f)\cap\operatorname{supp}(g).$$

The historical definition of local Lie algebra follows this example to call a $\mathbb{R}$-linear Lie algebra structure on the sections of some vector bundle $(\Gamma(A),[,])$ local when:

$$\forall a,b\in\Gamma(A) \ : \ \operatorname{supp}([a,b]) \subset \operatorname{supp}(a)\cap\operatorname{supp}(b).$$

The locality property of differential operators implies that this requirement is equivalent to the Lie bracket acting as a differential operator on each of its arguments.

This prompts us to give a more direct definition as follows: a vector bundle $A\to M$ is said to carry a local Lie algebra structure on its sections if $(\Gamma(A),[,])$ is a $\mathbb{R}$-linear Lie algebra bracket such that its adjoint map is a differential operator of the form

$$ad_{[,]}:\Gamma(A)\to \operatorname{Diff}_1(A).$$

$$ad_{[,]}\in \operatorname{Diff}_1(A,\operatorname{diff}_1(A)), \quad \operatorname{diff}_1(A):=J^1A^*\otimes A$$

The first obvious source of examples of local Lie algebras are differential operators themselves.

Exercise: Let $\Delta,\nabla\in\operatorname{Diff}_1(A)$ with symbol maps $\delta,\varrho\in\Gamma(T^*M\otimes A^*\otimes A)$, show that

$$[\Delta,\nabla](f\cdot s) = f\cdot[\Delta,\nabla](s) + \left([\delta(df),\nabla]+[\Delta,\varrho(df)]\right)(s)$$

Denote the second term by $\lambda_{\Delta\nabla}(df) : \Gamma(A) \to \Gamma(A)$. This is $C^\infty(M)$-linear in all entries by construction. According to the Leibniz characterisation of differential operators, $[\Delta, \nabla] \in \mathrm{Diff}_1(A) \Leftrightarrow \lambda_{\Delta\nabla}(df)$ is $C^\infty(M)$-linear for all $f \in C^\infty(M)$. Further show:

$$\lambda_{\Delta\nabla}(df)(g \cdot s) = g \cdot \lambda_{\Delta\nabla}(df)(s) + \Big([\delta(df), \varrho(dg)] + [\delta(dg), \varrho(df)] \Big)(s)$$

Without explicit use of the Leibniz characterisation of differential operators, this is found as the failure of general differential operators to close under commutator:

$$[[[\Delta, \nabla], f], g] = \Big[[\nabla, f], [\Delta, g]\Big] + \Big[[\nabla, g], [\Delta, f]\Big].$$

II DERIVATIVE LIE ALGEBRAS

From the exercise above we can see that examples of local Lie algebras of differential operators will appear as we identify classes of differential operators whose symbols commute, so that the λ map above is $C^\infty(M)$-linear in all arguments for all pairs of differential operators.

A natural class of such differential operators are those whose symbols are multiples of the identity on sections $\delta(df) \in \mu(C^\infty(M)) = C^\infty(M) \cdot \mathrm{id}_A \in \mathrm{End}(A)$, these were identified as the derivations of the vector bundle in Lecture 5.

In light of this, it is natural to define a subclass of local Lie algebras by demanding that the bracket acts as a vector bundle derivation:

a derivative Lie algebra structure on a vector bundle $A \to M$ is a $\mathbb{R}$-linear Lie algebra $(\Gamma(A), [,])$ such that its adjoint map is a differential operator of the form:

$$\mathrm{ad}_{[,]} : \Gamma(A) \to \mathrm{Der}(A)$$

$$\mathrm{ad}_{[,]} \in \mathrm{Diff}_1(A, DA).$$

Proposition 6.1

Let $E \to M$ be a general vector bundle, then its der bundle DE carries a canonical derivative Lie algebra structure.

proof. As per our definition of DE in lecture 5, it is clear that $\Gamma(DE)$ carries a $\mathbb{R}$-linear Lie bracket given by the commutator under the

isomorphism $\Gamma(DE) \cong Der(E) \subset End_{\mathbb{R}}(\Gamma(E))$. The Jacobi identity of the commutator ensures that $ad_{[,]}$ is a $\mathbb{R}$-linear map $ad_{[,]} : \Gamma(DE) \to End_{\mathbb{R}}(\Gamma(DE))$.
It can be directly checked that:

$D, D' \in Der(E)$, $X_D, X_{D'} \in \Gamma(TM)$: $[D, D'](f \cdot s) = f \cdot [D, D'] + [X_D, X_{D'}][f] \cdot s$

thus we see that $Der(E)$ is closed under the commutator bracket.
As a brief lemma, note that, by definition $Diff_0(E) = End_{C^\infty(M)}(\Gamma(E)) \subset Der(E)$, and, further, for all $\phi \in Diff_0(E)$, $D \in Der(E)$:

$$\phi \circ D, \; D \circ \phi \in Der(E)$$

Lastly, we check that $ad_{[,]}$ is a differential operator by writing:

$$\begin{aligned} ad_{[,]}(f \cdot D)(D') := [f \cdot D, D'] &= f DD' - D' f D \\ &= f DD' - [D', f] D - f D'D \\ &= f([D, D']) - \sigma_{D'}(df) D \\ &= f \cdot ad_{[,]}(D)(D') - (\sigma_{-}(df) \circ D)(D') \\ &= \left(f \cdot ad_{[,]}(D) - \sigma_{-}(df) \circ D\right)(D') \end{aligned}$$

where we identify $\lambda = -\sigma_{-}(\,) \circ - \in \Gamma(T^*M \otimes DE^* \otimes DE^* \otimes DE)$ as the symbol of the adjoint map (which is $C^\infty(M)$-linear from the fact that σ is $C^\infty(M)$-linear in all its arguments. //

III) THE SYMBOL - SQUIGGLE THEOREM

The derivative Lie algebra $(\Gamma(A), [,])$ acts as a derivation on each bracket argument, the fundamental formula manifesting the interaction of the Lie algebra and $C^\infty(M)$-module structure is the Leibniz formula:

$$[a, f \cdot b] = f \cdot [a, b] + \lambda_a[f] \cdot b \qquad \forall f \in C^\infty(M),\ a, b \in \Gamma(A)$$

where λ_a is a $C^\infty(M)$-derivation defined simply as the $C^\infty(M)$-factor of the symbol of the inner-derivations:

$$\sigma_{ad_{[,]}(a)} = \lambda_a \otimes id_{\Gamma(A)} .$$

Since $ad_{[,]}$ is assumed to be a differential operator we see that the map

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$$\lambda : \Gamma(A) \longrightarrow \Gamma(TM)$$
$$a \longmapsto \lambda_a$$

is a differential operator $\lambda \in \mathrm{Diff}_1(A, TM)$. We call this map the Symbol of the derivative Lie algebra $(\Gamma(A), [,])$. The symbol of this map will be a map of the form $\sigma_\lambda \in \Gamma(T^*M \otimes A^* \otimes TM)$, we denote this by $\Lambda^\#$ (for reasons that will be apparent below) and call it the squiggle of the derivative Lie algebra (since it is the "symbol of the symbol").

Proposition 6.2

Let $(\Gamma(A), [,])$ be a derivative Lie algebra with symbol λ and squiggle $\Lambda^\#$, then we have the symbol-squiggle identity:

$$[f\cdot a, g\cdot b] = fg\cdot[a,b] + f\lambda_a[g]\cdot b - g\lambda_b[f]\cdot a + \Lambda(df\otimes a, dg\otimes b)$$

with the following compatibility identities:

i) $\Lambda^\#(df\otimes a)[g]\cdot b = -\Lambda^\#(dg\otimes b)[f]\cdot a$, hence defining $\Lambda \in \Gamma(\Lambda^2(TM\otimes A)\otimes A)$.

ii) $\lambda_{[a,b]} = [\lambda_a, \lambda_b]$

iii) $\lambda_{f\cdot a} = f\cdot\lambda_a + \Lambda^\#(df\otimes a)$

iv) $[\lambda_a, \Lambda^\#(df\otimes b)] = \Lambda^\#(d\lambda_a[f]\otimes b + df\otimes[a,b])$

v) $\Lambda(df\otimes a, \Lambda(dg\otimes b, dh\otimes c)) + \text{cyclic} =$
$= \lambda_b[f]\cdot\Lambda(dh\otimes c, dg\otimes a) + \text{cyclic}$.

proof. (propositions 2.3.1 and 2.7.2 in my thesis) //

Proposition 6.3 (Symbol-Squiggle Theorem)

Let $A \to M$ be a vector bundle with $\Sigma \subset \Gamma(A)$ a submodule of spanning sections ($\Gamma(A) = C^\infty(M)\cdot\Sigma$), then the datum of a derivative Lie algebra $(\Gamma(A), [,])$ is equivalent to the following triple:

i) a $\mathbb{R}$-linear Lie bracket $(\Sigma, [,])$

ii) a $\mathbb{R}$-linear map $\lambda : \Gamma(A) \longrightarrow \Gamma(TM)$

iii) $\Lambda \in \Gamma(\Lambda^2(TM\otimes A)\otimes A)$, inducing a $C^\infty(M)$-linear map $\Lambda^\# : \Gamma(T^*M\otimes A) \to \Gamma(TM)$

Such that identities i) – v) in proposition 6.2 hold.

proof. (proposition 2.7.2 in my thesis) //

All the expressions above are greatly simplified in the case when $\lambda : \Gamma(A) \to \Gamma(TM)$ is $C^\infty(M)$-linear (i.e. a differential operator of order 0), since this means that its symbol vanishes:

$$\lambda \ C^\infty(M)\text{-linear} \iff \Lambda = 0$$

When this is the case, we call λ the anchor of the derivative Lie algebra $(\Gamma(A), [,])$.

(IV) THE RANK 1 DICHOTOMY

Proposition 6.4

Any local Lie algebra structure $(\Gamma(A), [,])$ on a vector bundle with 1-dimensional fibre, $rk(A) = 1$, is necessarily a derivative Lie algebra.

proof. this is a direct consequence of the fact that for a line bundle A, $rk(A) = 1$, we have:

$$\mathrm{Diff}_1(A) = \mathrm{Der}(A),$$

which is itself a direct consequence of the fact that the bundle of endomorphisms of a line bundle is canonically trivializable:

$$rk(A) = 1 \Rightarrow \mathrm{End}(A) \cong \mathbb{R}_M \text{ with } 0 \neq id_A \in \Gamma(\mathrm{End}(A)) \text{ the trivialising section.}$$ //

Proposition 6.5

The symbol of a derivative Lie algebra structure $(\Gamma(A), [,])$ on a vector bundle with fibre of dimension 2 or greater is necessarily an anchor.

proof. Use local sections to span two $C^\infty(M)$-linearly independent local sections and take

$$\lambda_{f\cdot a}[g]\cdot b - g\cdot\lambda_b[f]\cdot a + fg[a,b] = [f\cdot a, g\cdot b] = f\cdot\lambda_a[g]\cdot b - \lambda_{g\cdot b}[f]\cdot a + fg[a,b]$$

the $C^\infty(M)$-linear independence then implies that

$$\lambda_{f\cdot a}[g] = f\cdot\lambda_a[g] \quad \forall f, g \in C^\infty(M). \quad //$$

(V) IDENTIFYING CATEGORIES OF LOCAL LIE ALGEBRAS

We can see from Proposition 6.4 above that local Lie algebras on line bundles are a natural class of local Lie algebras, these are indeed known in the literature as Jacobi structures. In future lectures we will see how a "unit-free" generalisation of the category of smooth manifolds allows us to correctly identify the category of Jacobi structures.

Derivative Lie algebras of rank 2 or higher are precisely Lie algebroids, as defined in Lecture 3. There we saw how the category of Lie algebroids could be defined in different, contained ways without much technical difficulty.

More general categories of local Lie algebras are hard to identify in general (see the exercise in section (I) above). This is currently an open area of research.